<u>Factorization of Cyclic and Symmetric polynomials</u>

1. Definitions

(a) $y = f(x, y, z)$ is **cyclic** if $f(x, y, z) = f(y, z, x)$

$y = f(w, x, y, z)$ is **cyclic** if $f(w, x, y, z) = f(x, y, z, w)$

(b) A function in any number of variables is **symmetric** when it is unaltered by interchanging any two of the variables.

$y = f(x, y, z)$ is **symmetric** if $f(x, y, z) = f(y, x, z) = f(z, y, x)$

(c) A polynomial is **homogeneous** if the degree of each term is the same.

e.g. $f(x, y, z) = x^2y + y^2z + 2xyz$ is homogeneous, but $g(x, y, z) = x^2yz + x$ is not.

2. Table of cyclic and symmetric polynomials:

Homogeneous polynomials with variables x, y, z		
Degree	**Cyclic**	**Symmetric**
1	$\boldsymbol{A}(x + y + z)$	$\boldsymbol{A}(x + y + z)$
2	$\boldsymbol{A}(x^2 + y^2 + z^2) + \boldsymbol{B}(xy + yz + zx)$	$\boldsymbol{A}(x^2 + y^2 + z^2) + \boldsymbol{B}(xy + yz + zx)$
3	$\boldsymbol{A}(x^3 + y^3 + z^3) + \boldsymbol{B}(x^2y + y^2z + z^2x) + \boldsymbol{C}(xy^2 + yz^2 + zx^2) + \boldsymbol{D}xyz$	$\boldsymbol{A}(x^3 + y^3 + z^3) + \boldsymbol{B}(x^2y + y^2z + z^2x + xy^2 + yz^2 + zx^2) + \boldsymbol{C}xyz$

Homogeneous polynomials with variables w, x, y, z		
Degree	**Cyclic**	**Symmetric**
1	$\boldsymbol{A}(w + x + y + z)$	$\boldsymbol{A}(w + x + y + z)$
2	$\boldsymbol{A}(w^2 + x^2 + y^2 + z^2) + \boldsymbol{B}(wx + xy + yz + zw)$	$\boldsymbol{A}(w^2 + x^2 + y^2 + z^2) + \boldsymbol{B}(wx + wy + wz + xy + xz + yz)$
3	$\boldsymbol{A}(w^3 + x^3 + y^3 + z^3) + \boldsymbol{B}(w^2x + x^2y + y^2z + z^2x)$ $+ \boldsymbol{C}(wx^2 + xy^2 + yz^2 + zx^2) + \boldsymbol{D}(wxy + xyz + yzw + zwx)$	$\boldsymbol{A}(w^3 + x^3 + y^3 + z^3) + \boldsymbol{B}(w^2x + w^2y + w^2z + x^2y + x^2z + y^2z$ $wx^2 + wy^2 + wz^2 + xy^2 + xz^2 + yz^2) + \boldsymbol{C}(wxy + xyz + yzw + zwx)$

3. Useful Theorems

(a) The algebraic sum, difference, product and quotient of two cyclic (or symmetric) functions are cyclic (symmetric).

(b) Factor theorem

$(x - y)$ is a factor of $f(x, y, z) \Leftrightarrow f(y, y, z) = 0$

(c) A symmetric polynomial is cyclic but not vice versa.

For example, $f(x, y, z) = x^2y + y^2z + z^2x$ is cyclic but not symmetric.

4. An example

Factorize $f(x, y, z) = x^3 (y - z) + y^3 (z - x) + z^3 (x - y)$	
$f(y, z, x) = y^3 (z - x) + z^3 (x - y) + x^3 (y - z)$ $= f(x, y, z)$ $\therefore$ f is cyclic	Check that whether f is cyclic / symmetric: f is cyclic but not symmetric. f is homogeneous of degree 4.
$f(y, y, z) = y^3 (y - z) + y^3 (z - y) + z^3 (y - y)$ $= y^3 (y - z) - y^3 (y - z) + 0$ $= 0$ By Factor theorem, $(x - y)$ is a factor of f.	Sometimes, there is no need to expand all terms. Employ $(y - z)^{2n-1} = - (z - y)^{2n-1}$
Since f is cyclic, $(x - y)(y - z)(z - x)$ is a factor of f.	There is no need to use factor theorem again.
Since $\deg[f(x, y, z)] = 4$ $\deg[(x - y)(y - z)(z - x)] = 3$ $f(x, y, z) = (x - y)(y - z)(z - x)[k(x + y + z)]$	Product of cyclic polynomials is cyclic. We therefore need a cyclic polynomial of degree 1. Use the table in 2 above to help.
$f(2, 1, 0) = (2 - 1)(1 - 0) (0 - 1) [k(2 + 1 + 0)]$ $= - 6k$ $f(2, 1, 0) = 2^2(1 - 0) + 1^3(0 - 2) + 0^3 (2 - 1) = 6$ $\therefore - 6k = 6$ $\therefore k = - 1$	Instead of substitutions, you may also use **Comparing coefficient method**. Compare coefficients of x^3y term in this case can give $k = -1$.
$\therefore$ $f(x, y, z)$ $= - (x - y)(y - z)(z - x)(x + y + z)$	You may also write: $f(x, y, z) = (x + y + z)(x - z)(z - y)(y - z)$

5. Some common factors of cyclic polynomial $f(x, y, z)$:

Factor	**Test**
$(x + y)(y + z)(z + x)$	$f(-y, y, z) = 0$
$(x - y)(y - z)(z - x)$	$f(y, y, z) = 0$
xyz	$f(0, y, z) = 0$
$x + y + z$	$f(-y -z, y, z) = 0$
$(x + y - z)(y + z - x)(z + x - y)$	$f(z - y, y, z) = 0$